OptimizationCity

ENCYCLOPEDIA OF OPERATIONS RESEARCH

Course 2: Primal simplex method

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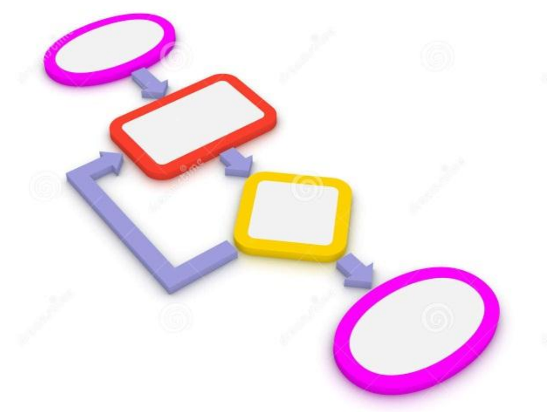

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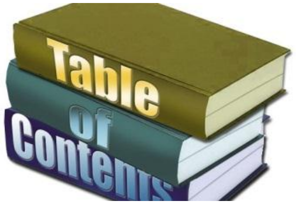


Basics of the primal simplex method

Basics of the primal simplex method

In an algebraic method, working with equal equations is far simpler than inequalities. Hence, the first step of the simplex method (omit the primal for the rest of manuscript) is to convert functional constraints from an unequal form to equal one. The constraints of negativity can be left unequal because they do not enter directly into the solution process. The conversion of an inequality to equality is done by introducing slack variables. Next, we solve the following model using Simplex method in the form of an example. The model is as follows:

$$(P1)\quad Max\ \ Z = 3x_1 + 5x_2$$

$$s.t.$$

$$(1)\quad x_1 \leq 4$$

$$(2)\quad 2x_2 \leq 12$$

$$(3)\quad 3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0, x_2 \geq 0.$$

Basics of the primal simplex method

$$(P1\!-\!1)\quad Max\ \ Z = 3x_1 + 5x_2$$

$s.t.$

$$
\begin{aligned}
&(1)\quad x_1 \qquad\quad + x_3 \qquad\qquad = 4\\
&(2)\qquad\quad 2x_2 \qquad + x_4 \qquad = 12\\
&(3)\quad 3x_1 + 2x_2 \qquad\qquad + x_5 = 18\\
&\qquad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Slack variables

Basics of the primal simplex method

Model (P1-1) is exactly the same as model (P1), but this new format is much simpler for algebraic operations. In model (P1-1) the number of variables is equal to 5 and the number of constraints is equal to 3, in which we are faced with a set of constriant with two degrees of freedom(number of variables-number of constriants). In this case, you can set the desired value for two additional variables in each step and solve the system equation of three constraint having three variables. In the simplex method, these two additional variables are set equal to zero. Variables that are considered zero are called **Non-basic variables** and others are called **Basic variables**. The solution that all non-basic variables are equal to zero is called the **Basic solution** and the basic solution in which the basic variables are non-negative is called the **Basic feasible solution**.

Basics of the primal simplex method

$(P1-2) \quad Max \ Z$

$s.t.$

$$
\begin{aligned}
&(0) Z - 3x_1 - 5x_2 &&= 0 \\
&(1) \quad x_1 + x_3 &&= 4 \\
&(2) \quad 2x_2 + x_4 &&= 12 \\
&(3) \quad 3x_1 + 2x_2 + x_5 &&= 18 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

The primal simplex method

In this section, we will present the formal form of simplex method. In short, at each step the simplex method seeks to find the basic feasible solution provided that this solution is not worse than the previous solution to finally find an optimal basic feasible solution. To convert from one basic feasible solution to another , it is sufficient to convert one basic variable to a non-basic variable (leaving variable) and a non-basic variable to a basic variable (entering variable). With these changes, the current basic feasible solution moves to the adjacent basic feasible solution. If a basic feasible solution is better than all the adjacent basic solution, that solution is the optimal solution and the algorithm ends at this point. The following figure summarizes the steps of the simplex method in general.

The primal simplex method

Initial step

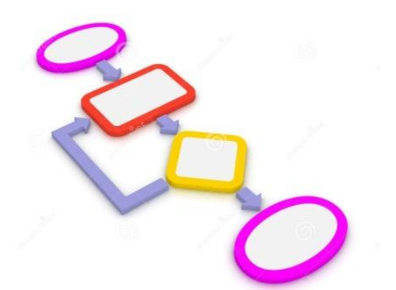
Find the feasible basic solution

Iterative step

Move to adjacent feasible basic solution

Stopping step

If there is no better adjacent solution, stop it, otherwise go to iterative step.



The primal simplex method

Intial step:

Call the slack variables (x_3, x_4, x_5) as the basic variables and set the (x_1, x_2) variables as non-basic variables equal to zero. For simplicity of calculation, the coefficients of the variables and the value on the right hand side are written in the table named the **simplex tableau**. The example simplex tableau (P1-2) is as follows.

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side
Z	0	1	-3	-5	0	0	0	0
X_3	1	0	1	0	1	0	0	4
X_4	2	0	0	2	0	1	0	12
X_5	3	0	3	2	0	0	1	18

Stoping step:

If and only if all coefficients of row zero are non-negative (≥ 0), then the current basic feasible solution is the optimal, and then stop. Otherwise, repeat the interative step to find the adjacent basic feasible solution.

The primal simplex method

Iterative step:

Substep 1: Select the variable that has the largest negative coefficient for non-basic variable in row zero as the **entering variable**. Increasing the value of this non-basic variable leads to the fastest growth rate of the objective function. Draw a rectangle around the column below entering variable and call **pivot column**. In this example, the largest negative coefficient (-5) is for the variable (x_2) and therefore it is selected as the entering variable.

Substep 2: The **leaving basic variable** is determined as follows.

A) Consider the positive coefficients of the pivote column

B) Divide the right hand side into positive coefficients

C) Select the row for which the ratio obtained in part (B) is the smallest.

D) The basic variable of this row is the leaving basic variable.

Substep 3: Draw a rectangle around this row and call **pivot row**. The value in both rectangles (pivote column and pivote row) is called the **pivot number**.

The results of the above operations are given in the table below.

The primal simplex method

Input variable

Basic variable (0)	Row (1)	Z (2)	X_1 (3)	X_2 (4)	X_3 (5)	X_4 (6)	X_5 (7)	RHS (8)	Ratio test
Z	0	1	-3	-5	0	0	0	0	
X_3	1	0	1	0	1	0	0	4	Min 6=12/2
X_4	2	0	0	2	0	1	0	12	9=18/2
X_5	3	0	3	2	0	0	1	18	

Output variable

Pivot row

Pivot column

Pivot number

In the above table, the variable x_2 is the entering variable and the variable x_4 is the leaving variable.

The primal simplex method

Substep 4: Get the new basic solution with the help of a new simplex tableau. The steps to obtain a new tableau are as follows.

A) In column (0), delete the leaving variable, x_4, and replace it with the entering variable, x_2,.

B) To convert the coefficient of entering variable to **one**, divide the pivote row by the pivote number.

C) In order to remove the e basic variable from the other row, each row (even row zero) except the pivote row should be changed as follows.

C1. Multiply the pivote row by a nonzero constant which lead to become zero the coefficient of pivote column except from pivote number

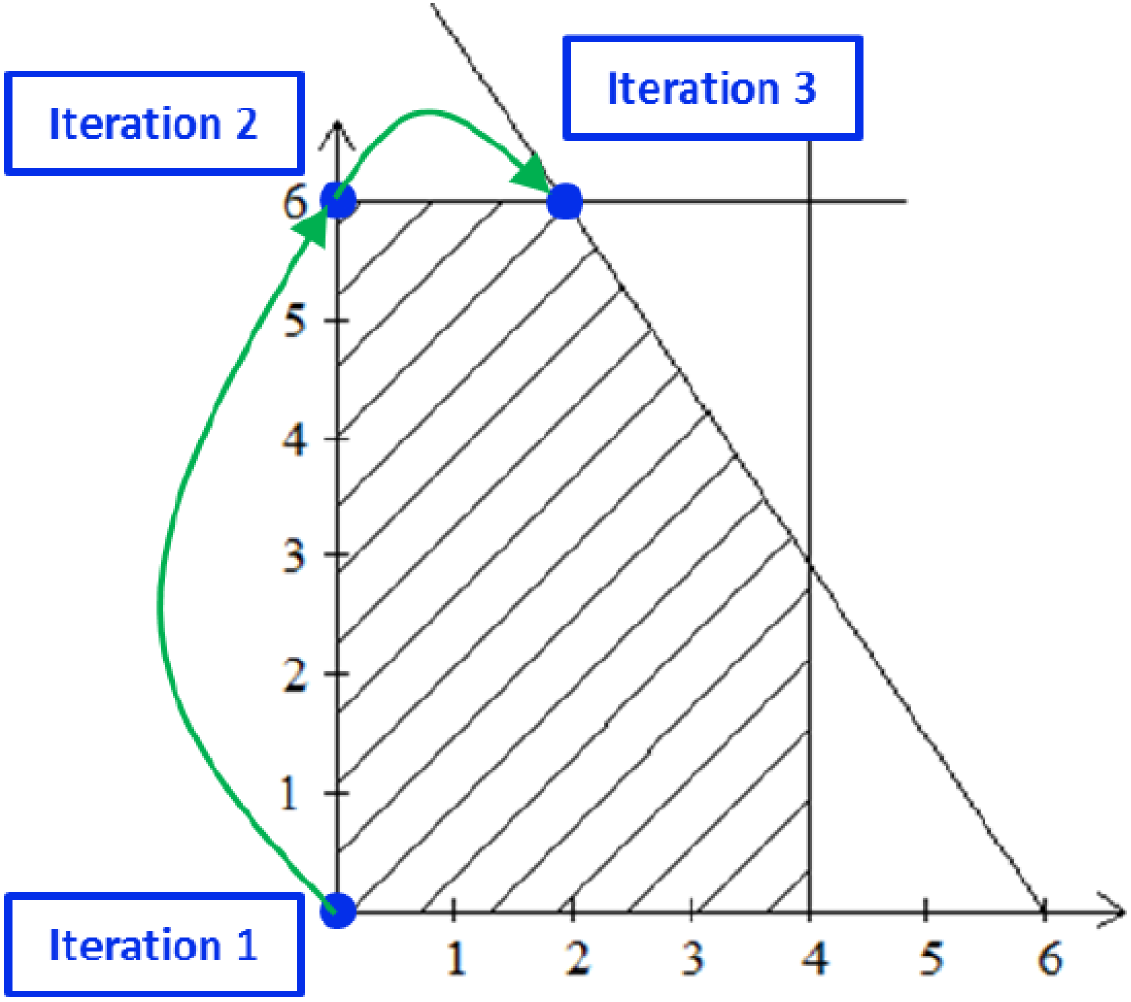
C2. Add a multiple of pivote row to another row.

After creating a new simplex tableau, we go to the stopping step. If the stopping condition is met, the algorithm will stop; otherwise we will go to iterative step. We will continue this process until the stopping condition is met. The complete table of samples for (P2-1) is as follows.

The primal simplex method

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side	Ratio test
Z	0	1	-3	-5	0	0	0	0	
X_3	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	Min 6=12/2
X_5	3	0	3	2	0	0	1	18	9=18/2
Z	0	1	-3	0	0	2.5	0	30	
X_3	1	0	1	0	1	0	0	4	4=4/1
X_2	2	0	0	1	0	0.5	0	6	
X_5	3	0	3	0	0	-1	1	6	Min 2=6/3
Z	0	1	0	0	0	1.5	1	36	
X_3	1	0	0	0	1	0.333	-0.333	2	
X_2	2	0	0	1	0	0.5	0	6	
X_1	3	0	1	0	0	-0.333	0.333	2	

The primal simplex method



Shadow price

The simplex method produces other valuable information in addition to the optimal solution. The shadow value of the source i (denoted by y_i^*) measures the marginal value of source i, which indicates the rate of increase of Z due to a slight increase in the right hand side of source i (b_i). Note that the rate of increase must be small enough that the current set of variables remains optimal because as soon as the set of basic variables changes, the shadow price also changes. The coefficient of i-th slack variable, which is related to i-th constraint, in row zero of final simplex tabluea determines the shadow price of i-th constraint.

Special cases in simplex method

Picking the entering variable

Suppose two or more non-basic variables have the largest negative coefficient. Here, pick the one with the largerst index. For example, consider (P1) model with different objective function, $Z=3x_1+3x_2$. Both x_1 and x_2 had a coefficient of 3, and 3; we pick x_2 .

Special cases in simplex method

Degeneracy

A linear model is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is a problem in practice, because it makes cycling in the basic solution and then makes the simplex algorithm slower.

Bland’s rule

Unbounded Z

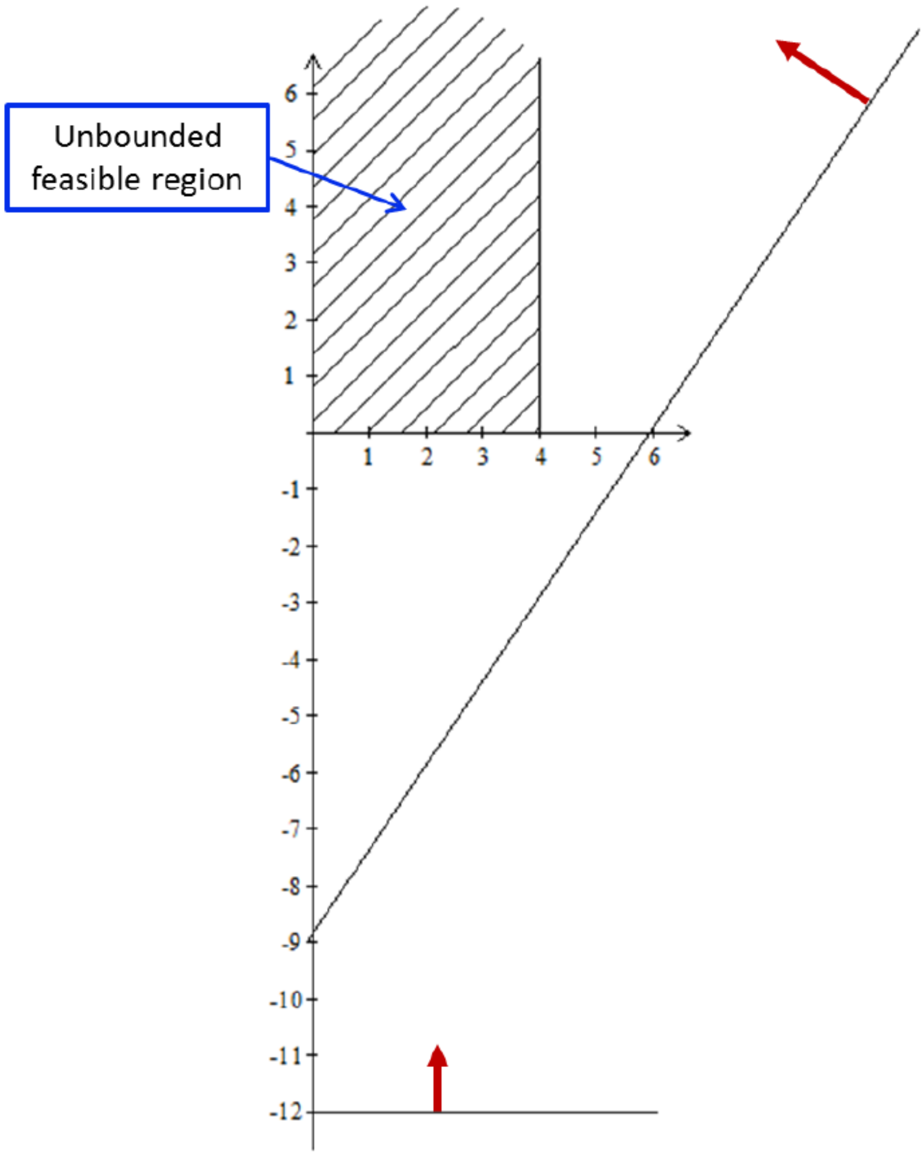
Consider a situation where none of the basic variables have leaving conditions, and the value of the basic entering variables can increase infinitely without any of the other basic variables being negative. This condition occurs when coefficients of the pivote column in the simplex table are all negative or zero (non-positive). Consider the following model to clarify the issue.

$$Max\ Z = 3x_1 + 5x_2$$

$s.t.$

$$\begin{aligned}
&(1) \quad x_1 \le 4 \\
&(2) \quad -x_2 \le 12 \\
&(3) \quad 3x_1 - 2x_2 \le 18 \\
&\quad\quad x_1 \ge 0, x_2 \ge 0.
\end{aligned}$$

Special cases in simplex method



Special cases in simplex method

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side
Z	0	1	-3	-5	0	0	0	0
X_3	1	0	1	0	1	0	0	4
X_4	2	0	0	-2	0	1	0	12
X_5	3	0	3	-2	0	0	1	18

$$x_4 = 12 + 2x_2$$

$$x_5 = 18 + 2x_2 - 3x_1$$

Special cases in simplex method

Multiple optimal solutions

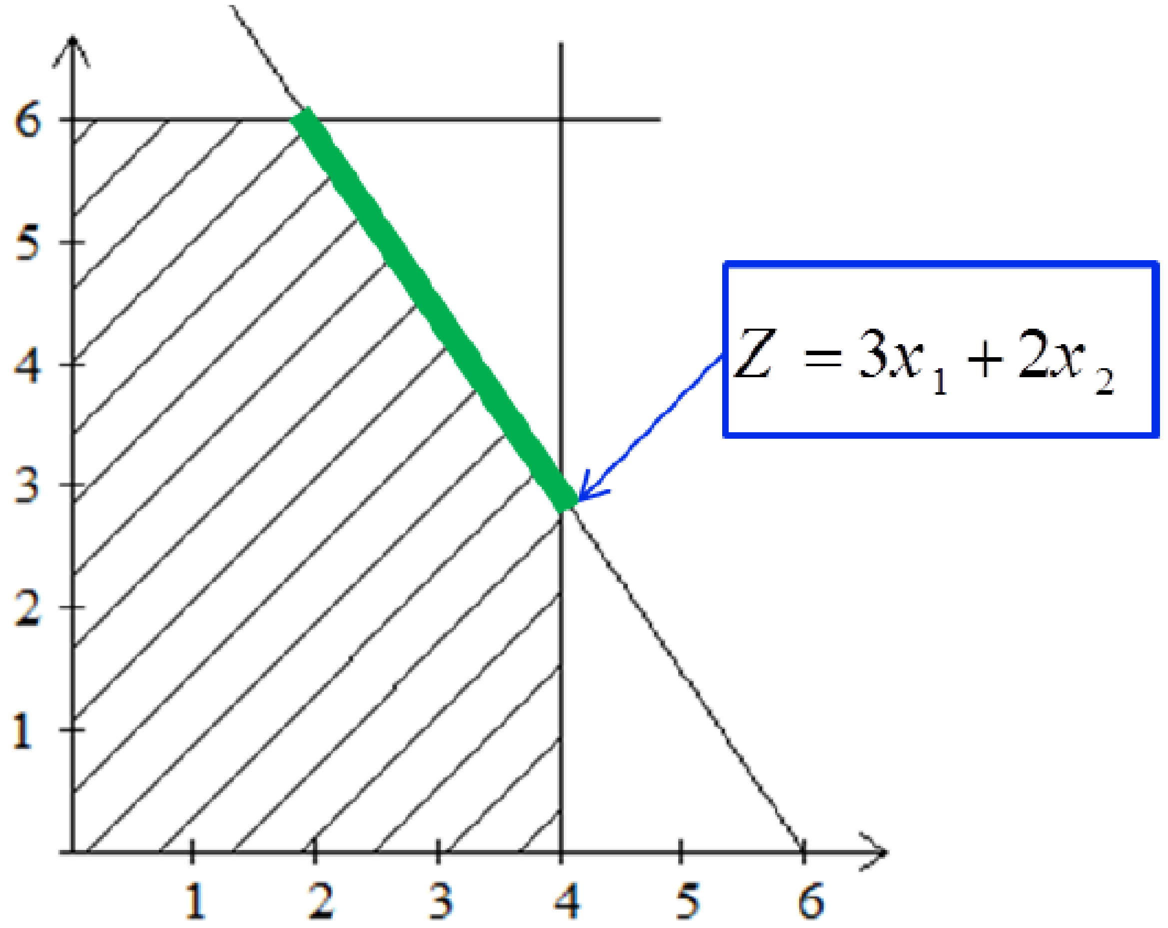
Whenever a problem has more than one optimal basic feasible solution, at least one of the nonbasic variables has a coefficient of zero in the final row zero, so increasing any such variable will not change the value of Z.

$$Max\ Z = 3x_1 + 2x_2$$

$s.t.$

$$\begin{aligned}
&(1) \quad x_1 \leq 4\\
&(2) \quad 2x_2 \leq 12\\
&(3) \quad 3x_1 + 2x_2 \leq 18\\
&\quad\quad x_1 \geq 0, x_2 \geq 0.
\end{aligned}$$

Special cases in simplex method



Special cases in simplex method

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side	
Z	0	1	-3	-2	0	0	0	0	Non optimal
X_3	1	0	1	0	1	0	0	4	Min 4=4/1
X_4	2	0	0	2	0	1	0	12	
X_5	3	0	3	2	0	0	1	18	6=18/3
Z	0	1	0	-2	3	0	0	12	Non optimal
X_1	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	6=12/2
X_5	3	0	0	2	-3	0	1	6	Min 3=6/2
Z	0	1	0	0	0	0	1	18	**Optimal**
X_1	1	0	1	0	1	0	0	4	4=4/1
X_4	2	0	0	0	3	1	-1	6	Min 2=6/3
X_2	3	0	0	1	-1.5	0	0.5	3	
Z	0	1	0	0	0	0	1	18	**Optimal**
X_1	1	0	1	0	0	-0.333	0.333	2	
X_3	2	0	0	0	1	0.333	-0.333	2	
X_2	3	0	0	1	0	0.5	0	6	

Special cases in simplex method

Equality constriants

Any equality constraint

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$$

actually is equivalent to a pair of inequality constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$$

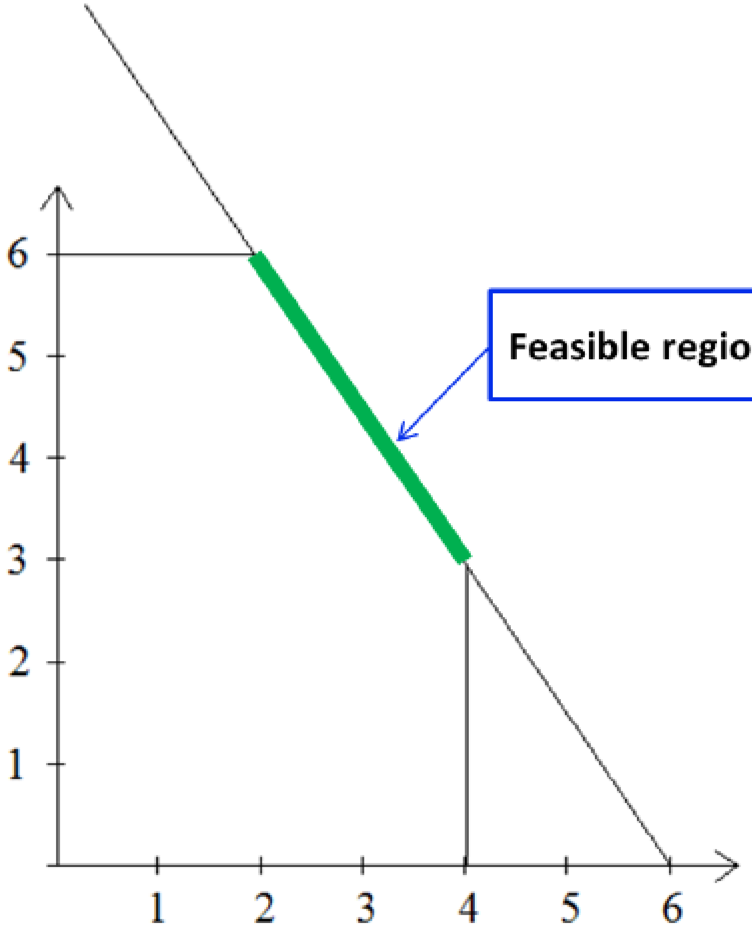
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$$

Special cases in simplex method

$$\text{Max } Z$$

$s.t.$

$$\begin{aligned}
&(0)\ Z - 3x_1 - 5x_2 &&= 0 \\
&(1)\ x_1 + x_3 &&= 4 \\
&(2)\ 2x_2 + x_4 &&= 12 \\
&(3)\ 3x_1 + 2x_2 &&= 18 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$



Special cases in simplex method

1. Apply the **artificial variable** by introducing a nonnegative artificial variable (call it $\bar{x}_5$) into constraint (3), then we have:

$$3x_1 + 2x_2 + \bar{x}_5 = 18$$

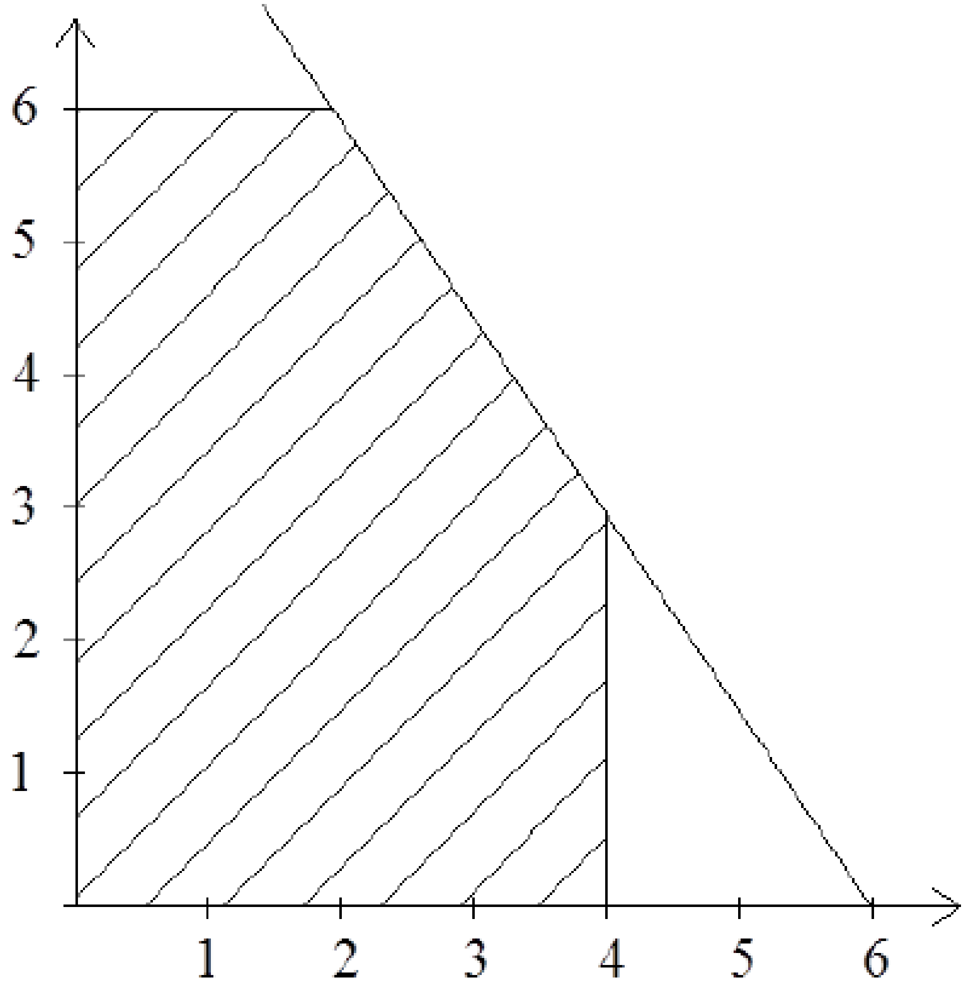
And then the model would be:

$$Max\ Z$$

$$s.t.$$

$$\begin{aligned}
&(0) Z - 3x_1 - 5x_2 &&= 0\\
&(1) \quad x_1 + x_3 &&= 4\\
&(2) \quad 2x_2 + x_4 &&= 12\\
&(3) \quad 3x_1 + 2x_2 + \bar{x}_5 &&= 18\\
&\quad x_1, x_2, x_3, x_4, \bar{x}_5 \geq 0
\end{aligned}$$

Special cases in simplex method



Special cases in simplex method

2. Assign the penalty to having $\bar{x}_5 = 0$ by changing the objective function $Z - 3x_1 - 5x_2 + M\bar{x}_5 = 0$, where M symbolically represents a huge positive number. (This method of forcing $\bar{x}_5 = 0$ in the optimal solution is called the **Big M method**.)

Converting Equation (0) to Proper Form.

					$\bar{x}_5$		
Row 0	-3	-5	0	0	M	0	Ⓐ
Row 3	3	2	0	0	1	18	Ⓑ

Row 0	(-3M-3)	(-2M-5)	0	0	0	-18M	Ⓐ+(-M)Ⓑ

Special cases in simplex method

Basic variable	Row	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	Right Hand Side	
Z	0	1	-3M-3	-2M-5	0	0	0	-18M	Non optimal
X_3	1	0	1	0	1	0	0	4	Min 4=4/1
X_4	2	0	0	2	0	1	0	12	
$\bar{X}_5$	3	0	3	2	0	0	1	18	6=18/3
Z	0	1	0	-2M-5	3M+3	0	0	-6M+12	Non optimal
X_1	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	6=12/2
$\bar{X}_5$	3	0	0	2	-3	0	1	6	Min 3=6/2
Z	0	1	0	0	-4.5	0	M+2.5	27	Non optimal
X_1	1	0	1	0	1	0	0	4	4=4/1
X_4	2	0	0	0	3	1	-1	6	Min 2=6/3
X_2	3	0	0	1	-1.5	0	0.5	3	
Z	0	1	0	0	0	1.5	M+1	36	**Optimal**
X_1	1	0	1	0	0	-0.333	0.333	2	
X_3	2	0	0	0	1	0.333	-0.333	2	
X_2	3	0	0	1	0	0.5	0	6	

In the above table, the optimal solution is (2,6,2,0,0).

Two-phase method

Two-phase method

the big M method → computational error.

phase 1 seeks to minimize the sum of artificial variables.

phase 2 uses the initial basic solution to proceed the simplex method.

Two-phase method

Example: Using the two-phase method, work through the simplex method step by step to solve the problem.

$$Max\ Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$$

$$s.t.$$

$$(1)\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 = 300$$

$$(2)\quad 8x_1 + x_2 + x_3 + 5x_4 = 300$$

$$x_i \geq 0 \quad i = 1,\dots,4.$$

Two-phase method

Solution:

Write the model in the standard form:

$$Max\ Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$$

$$s.t.$$

$$(1)\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 + x_5 \qquad = 300$$

$$(2)\quad 8x_1 + x_2 + x_3 + 5x_4 \qquad + x_6 = 300$$

$$x_i \geq 0 \quad i = 1, \dots, 6.$$

$$Min\quad w = x_5 + x_6$$

Two-phase method

$$Min\,W = x_5 + x_6$$

$s.t.$

$$(1)\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 + x_5 = 300$$

$$(2)\quad 8x_1 + x_2 + x_3 + 5x_4 + x_6 = 300$$

$$x_i \geq 0 \quad i = 1, \dots, 6.$$

Two-phase method

Basic variable	Row	Z	W	X_1	X_2	X_3	X_4	X_5	X_6	RHS	Note
W		0	-1	0	0	0	0	1	1	0	
Z	0	1	0	4	2	3	5	0	0	0	
X_5	1	0	0	2	3	4	2	1	0	300	
X_6	2	0	0	8	1	1	5	0	1	300	
W		0	-1	-10	-4	-5	-7	0	0	-600	Non optimal – Start phase 1
Z	0	1	0	-4	-2	-3	-5	0	0	0	
X_5	1	0	0	2	3	4	2	1	0	300	300/2=150
← X_6	2	0	0	8	1	1	5	0	1	300	**Min 300/8=75**
W		0	-1	0	-2.75	-3.75	-0.75	0	1.25	-225	Non optimal – Continue phase 1
Z	0	1	0	0	-1.5	-2.5	-2.5	0	0.5	150	
← X_5	1	0	0	1	2.75	3.75	0.75	1	-0.25	225	**Min 60=225/3.75**
X_1	2	0	0	0	0.125	0.125	0.625	0	0.125	375	3000=375/0.125
W		0	-1	0	0	0	0	1	1	0	**Optimal – End phase 1**
Z	0	1	0	0	0.333	0	-2	0.666	0.333	300	
X_3	1	0	0	0	0.733	1	0.2	0.266	-0.067	60	
X_1	2	0	0	1	0.033	0	0.6	-0.033	0.133	30	
Z	0	1	0	0	0.333	0	-2			300	Non optimal – Start phase 2
X_3	1	0	0	0	0.733	1	0.2			60	300=60/0.2
← X_1	2	0	0	1	0.033	0	0.6			30	**Min 50=30/0.6**
Z	0	1	0	3.33	0.44	0	0			400	**Optimal – End phase 2**
X_3	1	0	0	-0.333	0.722	1	0			50	
X_4	2	0	0	1.666	1.666	0	1			50	

The optimal solution of original model is $(0,0,50,50)$ and $Z^* = 400$.

Revised simplex method

Another practical method for solving linear programming problems is the revised simplex method. Although the simplex method is suitable for performing manual calculations, it does not have the necessary efficiency to solve large problems by computer. The reason can be found in the storage of information that may never be used in simplex iterations. For example, some variables never meet the necessary conditions to be selected as input variables, as a result, all calculations related to the coefficients of these variables in the objective function and constraints will remain unused. In fact, the column corresponding to that variable is calculated, but it is not actually used.

Revised simplex method

Matrix representation of the linear programming

In general, the standard linear programming problem is represented as follows:

$$Max\ Z = cx$$

$$Ax \leq \bar{b}$$

$$x \geq 0$$

Where

$$x = [x_1, x_2, \ldots, x_n]^T \quad c = [c_1, c_2, \ldots, c_n]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \bar{b} = [\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m]^T$$

Revised simplex method

Example:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 3x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

In the above model:

$$x = [x_1, x_2, x_3]^T \quad c = [4, 3, 6]$$

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix} \qquad \bar{b} = [30, 40]^T$$

$$\rightarrow Max\ Z = [4, 3, 6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Revised simplex method

$$\begin{aligned} &Max\ Z = cx \\ &Ax + Is = \bar{b} \\ &x, s \geq 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} &Max\ Z = cx \\ &[A, I]\begin{bmatrix} x \\ s \end{bmatrix} = \bar{b} \\ &x, s \geq 0 \end{aligned}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

where I is called the m-by-m unit matrix and

$$s = [s_1, s_2, \ldots, s_m]^T$$

Revised simplex method

$$Max\ Z = [4,3,6,0,0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{bmatrix}}_{[A,I]}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Revised simplex method

Example: Solve the following model using the revised simplex method.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$

$$3x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 3x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0
s_1	1	0	3	1	3	1	0	30
s_2	2	0	2	2	3	0	1	40

$\bar{a}_1$ ↗ (the x_1 column: 3, 2) B ↗ (the s_1, s_2 columns) $\bar{b}$ ↗ (the RHS column: 30, 40)

$$x_B = [s_1, s_2], x_N = [x_1, x_2, x_3]$$

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix}$$

Revised simplex method

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	2	-1	0	2	0	60
x_3	1	0	1	$\frac{1}{3}$	1	$\frac{1}{3}$	0	10
s_2	2	0	-1	1	0	-1	1	10

B^{-1}

$$x_B = [x_3, s_2], x_N = [x_1, x_2, s_1]$$

$$B = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$

$$3x_1 + x_2 + \boxed{3}x_3 \le 30$$

$$2x_1 + 2x_2 + \boxed{3}x_3 \le 40$$

$$x_1, x_2, x_3 \ge 0$$

Revised simplex method

The final simplex table is as follows.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	1	0	0	1	1	70
x_3	1	0	$\frac{4}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
x_2	2	0	-1	1	0	-1	1	10

B^{-1}

Where

$$x_B = [x_3, x_2], x_N = [x_1, s_1, s_2]$$

$$B = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Revised simplex method

$$[A, I]\begin{bmatrix} x \\ s \end{bmatrix} = \bar{b}$$

$$[B, N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = \bar{b} \rightarrow Bx_B + Nx_N = \bar{b}$$

$$B^{-1}Bx_B + B^{-1}Nx_N = B^{-1}\bar{b}$$

$$x_B = B^{-1}\bar{b} - B^{-1}Nx_N$$

$$x_B = B^{-1}\bar{b}$$

Revised simplex method

The objective function will be as follows by separating the variables into basic and non-basic variables:

$$Z = c_B x_B + c_N x_N$$

$$Z = c_B \left(B^{-1}\overline{b} - B^{-1} N x_N \right) + c_N x_N$$

$$Z = c_B B^{-1}\overline{b} - c_B B^{-1} N x_N + c_N x_N$$

$$Z = c_B B^{-1}\overline{b} - \left(c_B B^{-1} N - c_N \right) x_N$$

Since x_N represent non-basic variables and have zero value, we have as a result:

$$Z = c_B B^{-1}\overline{b}$$

Revised simplex method

The coefficient of the non-basic variables in the objective function, that is, the coefficient x_N, which is denoted by the symbol z_j-c_j, will be as follows according to the above relationships:

$$z_j - c_j = c_B B^{-1} N - c_N$$

Coefficients of non-essential variables (x_N) in the constraints, according to the following equation

$$x_B = B^{-1}\overline{b} - B^{-1} N x_N$$

Equal to

$$B^{-1} N$$

Revised simplex method

$$N = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n]$$

where each column of N shows the variable coefficients x_j in the constraints. In this way, the following relationship is used to calculate the pivote column numbers:

$$a_j = B^{-1}\bar{a}_j$$

Revised simplex method

Example:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$

$$3x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 3x_3 \le 40$$

$$x_1, x_2, x_3 \ge 0$$

Revised simplex method

$$c_B = \begin{bmatrix} \overset{x_3}{6}, \overset{x_2}{3} \end{bmatrix}, c_N = \begin{bmatrix} \overset{x_1}{4}, \overset{s_1}{0}, \overset{s_2}{0} \end{bmatrix}, B = \begin{bmatrix} \overset{x_3}{3} & \overset{x_2}{1} \\ 3 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix}$$

$$x_N = [x_1, s_1, s_2], x_B = [x_3, x_2]$$

$$z_j - c_j = c_B B^{-1} N - c_N = \begin{bmatrix} \overset{x_3}{6}, \overset{x_2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \overset{x_1}{3} & \overset{s_1}{1} & \overset{s_2}{0} \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \overset{x_1}{4}, \overset{s_1}{0}, \overset{s_2}{0} \end{bmatrix} = \begin{bmatrix} \overset{x_1}{1}, \overset{s_1}{1}, \overset{s_2}{1} \end{bmatrix}$$

Revised simplex method

$$x_B = B^{-1}\overline{b} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} \frac{20}{3} \\ 10 \end{bmatrix}$$

$$Z = c_B B^{-1}\overline{b} = [6,3] \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 70$$

The calculation of the x_1 variable column is done as follows:

$$a_1 = B^{-1}\overline{a_1}$$

$$a_1 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}$$

Revised simplex method

Revised simplex algorithm

Step 1: Determine the input variable. At each iteration, find the non-prime coefficients in the objective function using the $c_B B^{-1} N - c_N$. If these values are all non-negative, you have reached the **optimal solution**. Calculate the optimal solution using the following relations.

$$x_B = B^{-1}\overline{b}$$

$$Z = c_B B^{-1}\overline{b}$$

Otherwise, select the variable with the most negative calculated coefficient. This variable is the **input variable**.

Revised simplex method

Step 2: Determine the output variable. Selecting the **output variable** requires having the coefficients of the input variable in the constraints and numbers on the right hand side. If the variable x_j is the input variable, its coefficients in the constraints are obtained from the following relationship.

$$a_j = B^{-1}\overline{a}_j$$

The value of the current basic variables which is equal to numbers on the right hand side is:

$$x_B = B^{-1}\overline{b}$$

The basic output variable is obtained as follows.

$$\underset{j}{Min}\left\{\frac{x_B}{a_j}\right\}$$

Revised simplex method

Step 3: Calculate the new B^{-1} and determine **the new basic variable** and go to step 1.

Revised simplex method

Example: Consider the following model. Obtain the optimal solution using the revised simplex algorithm.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$

$$3x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 3x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0$$

Revised simplex method

Solution:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$

$$3x_1 + x_2 + 3x_3 + s_1 = 30$$

$$2x_1 + 2x_2 + 3x_3 + s_2 = 40$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Revised simplex method

Iteration 1

Step 1. Determine the input variable. The basic variable of iteration 1, s_1 and s_2 and their coefficient in the objective function is zero ($c_B=(0,0)$). The coefficients of non-basic variables ($x_N=(x_1,x_2,x_3)^T$) are calculated as follows:

$$z_j - c_j = c_B B^{-1} N - c_N = \left[\overset{s_1\ \ s_2}{0,0}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\overset{x_1}{3} & \overset{x_2}{1} & \overset{x_3}{3}\\ 2 & 2 & 3\end{bmatrix} - \left[\overset{x_1}{4},\overset{x_2}{3},\overset{x_3}{6}\right] = \left[\overset{x_1}{-4},\overset{x_2}{-3},\overset{x_3}{-6}\right]$$

Because the coefficient of x_3 is equal to -6 and the most negative number, this is the input variable shown in the table below.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0

$$-6 = \min\{-4,-3,-6\}$$

Step 2. Determine the output variable. The x_3 coefficients in the constraints are calculated as follows:

$$a_3 = B^{-1}\overline{a}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

$$\rightarrow Min \left\{ \frac{30}{3}, \frac{40}{3} \right\} = 10$$

Because the smallest number of the result corresponds to the basic variable s_1, this variable is selected as the output variable. The calculations of steps 1 and 2 are summarized as follows:

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0
s_1	1				3			30
s_2	2				3			40

Revised simplex method

Step 3.

$$x_B = [x_3, s_2]^T, c_B = [6, 0] \rightarrow B = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix}$$

Iteration 2

Step 1. Determine the input variable. The coefficients of non-basic variables, $x_N=[x_1,x_2,s_1]$, are calculated as follows.

$$z_j - c_j = c_B B^{-1} N - c_N = \begin{bmatrix} \overset{x_3}{6}, \overset{s_2}{0} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \overset{x_1}{3} & \overset{x_2}{1} & \overset{s_1}{1} \\ 2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} \overset{x_1}{4}, \overset{x_2}{3}, \overset{s_1}{0} \end{bmatrix} = \begin{bmatrix} \overset{x_1}{2}, \underset{\Uparrow}{\overset{x_2}{-1}}, \overset{s_1}{2} \end{bmatrix}$$

x_2 becomes the input variable.

Revised simplex method

Step 2. Determine the output variable.

$$a_2 = B^{-1}\overline{a}_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_3 \\ s_2 \end{bmatrix} = B^{-1}\overline{b} = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\rightarrow Min \left\{ \overset{x_3}{\frac{10}{\frac{1}{3}}}, \overset{s_2}{\frac{10}{\underset{\Uparrow}{1}}} \right\} = 10$$

s_2 is the output variable.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1		-1				
x_3	1	0		$\frac{1}{3}$				10
s_2	2	0		1				10

Revised simplex method

Step 3. Since x_2 is the input variable and s_2 is the output variable, the basic variables of the next iteration are:

$$x_B = [x_3, x_2]^T, c_B = [6, 3]$$

$$B = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix}$$

Revised simplex method

Iteration 3

Step 1. Determine the input variable. The coefficients of non-basic variables ($x_N=[x_1,s_1,s_2]^T$) are calculated as follows:

$$z_j - c_j = c_B B^{-1} N - c_N = \begin{bmatrix} \overset{x_3}{6}, \overset{x_2}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \overset{x_1}{3} & \overset{s_1}{1} & \overset{s_2}{0} \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \overset{x_1}{4}, \overset{s_1}{0}, \overset{s_2}{0} \end{bmatrix} = \begin{bmatrix} \overset{x_1}{1}, \overset{s_1}{1}, \overset{s_2}{1} \end{bmatrix}$$

Simplex method with bounded variable

Simplex method with bounded variable

In many linear programming problems, we encounter cases where the decision variables have a specific upper or lower limit. In this case, the form of the problem will be as follows:

$$Max\ Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, \ldots, m$$

$$x_j \le u_j, x_j \ge l_j \quad j = 1, \ldots, n$$

u_j and l_j are fixed numbers that show the maximum increase (upper limit) and the maximum decrease (lower limit) of variable x_j, respectively.

Exercises

Thanks

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