Fourth course:

Parametric linear programming

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Sensitivity analysis

Sensitivity analysis is a procedure that is implemented after obtaining the optimal solution. Sensitivity analysis determines the sensitivity of the optimal solution against certain changes in the original model.

As mentioned before, one of the assumptions of linear programming is that the parameters of the model including a_{ij}, c_j and b_i are certain and definite values. But we know that the value of each parameter used in the model is estimated based on assumptions and predictions. These estimates are based on information that is usually incomplete and sometimes non-existent. Therefore, the parameters that are first entered in formulating the model are considered only as an empirical estimate, and sometimes it is possible that people estimate the value of the parameters lower or higher than their actual value. the experienced manager always look at the results with skepticism. It even often looks at these results as a starting point for a comprehensive analysis.

It is for the above reasons that sensitivity analysis becomes important. Changes that are usually studied in the linear programming model include the following:

1) Changes in the numbers on the right hand side

2) Changes in the coefficients of the objective function

3) Add a new constraint

Generally, the result of these changes is one of the following three situations:

1) The optimal answer remains unchanged, that is, the basic variables and their values do not change at all.

2) Basic variables should not change, but their values should change.

3) The basic solution should be completely changed.

The first case shows the insensitivity of the optimal solution to model changes and the third case shows more sensitivity.

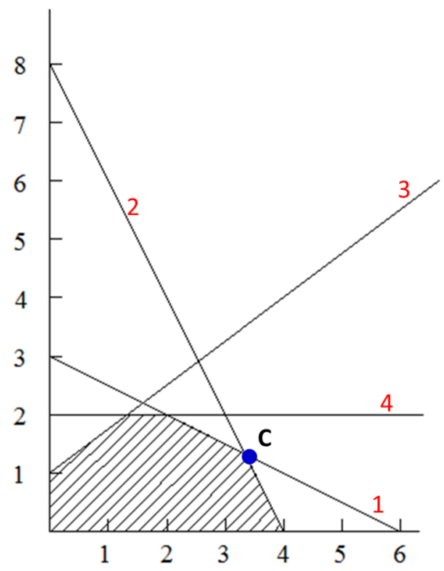
Sensitivity analysis of right hand side

The goal in analyzing the sensitivity of the numbers on the right hand side to determine a change in the range of the right hand side so that the final table will remain feasible (that is, none of the numbers on the right hand side will be negative). This case is explained using the following example.

Example:

In the following problem, in order to produce two products whose production values are represented by x_1 and x_2, four resources a, b, c, and d are used, which are on the right hand side of the constraint 1, 2, 3, and 4, respectively. The amount of available resources is equal to a=6, b=8, c=1, d=2, and the four constriants of the problem are written based on the limitations of these four resources. If the profit per unit of the first and second product is 3 and 2 respectively, the model is as follows.

$$
\begin{aligned}
Max \quad Z &= 3x_1 + 2x_2 \\
x_1 + 2x_2 &\le 6 \\
2x_1 + x_2 &\le 8 \\
-x_1 + x_2 &\le 1 \\
x_2 &\le 2 \\
x_1, x_2 &\ge 0
\end{aligned}
$$



The optimal solution of the problem, which is determined by point C, is:

$$x_1^* = 3\frac{1}{3}, x_2^* = 1\frac{1}{3}, s_1^* = 0, s_2^* = 0, s_3^* = 3, s_4^* = \frac{2}{3}$$

As can be seen, for production at the optimal point, sources 1 and 2 are completely finished, but the level of sources 3 and 4 will be equal to 3 and 0.666, respectively. In this example, the following question is raised.

How much can the amount of resources (numbers on the right) decrease or increase?

After determining the optimal solution, it will be possible to study the possible changes in the optimal solution. In particular, we are interested in two types of analysis:

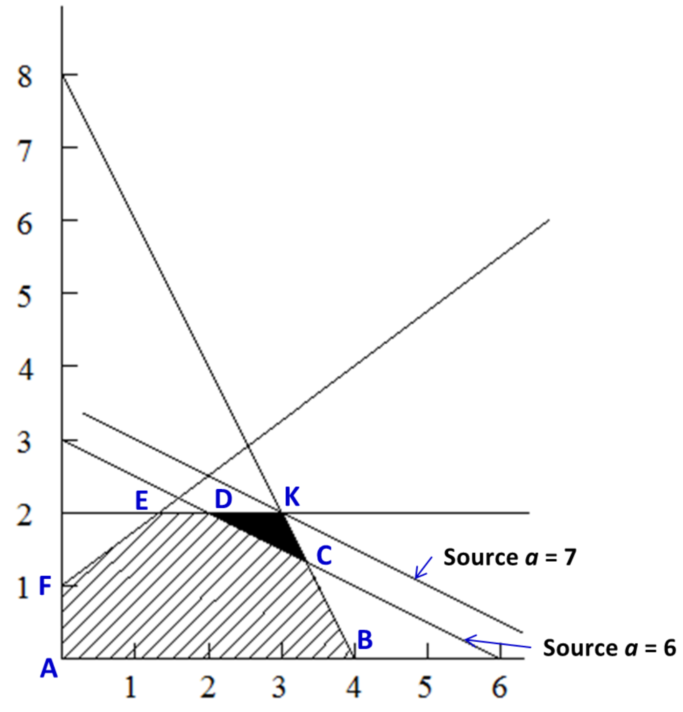
1) In order to improve the optimal value of the objective function, how much of a resource can be increased?

2) How much of a resource can be reduced without causing a change in the current optimal solution?

Since the source level is expressed by the numbers on the right side of the constraints, this type of analysis is called the s**ensitivity analysis of the right hand side**.

In the above problem, only constraints 1 and 2, which represent the amount of resources a and b, are effective. Logically, if a constraint is important, it is considered as a scarce resource because all available resources are completely used (s_1=0, s_2=0). On the other hand, a constraint indicates a non-scarce resource ($s_3 \neq 0, s_4 \neq 0$). Therefore, we are interested in knowing how much scarce resources increase, in order to improve the value of the objective function, and similarly, we want to change how much non-scarce resources can be reduced without affecting the optimal solution.

Constraints 1 and 2 represent scarce resources. First, source *a* is checked. As the amount of source *a* increases, constraint 1 or line CD moves upward and parallel to itself, and triangle CDK gradually shrinks (sides CK and DK of triangle DKC are constraints 2 and 4 .). When it reaches the point K, constraints 2 and 4 become important and the optimal solution is located at the point K and the ABKEF area is the feasible region.



The further increase of the first source capacity causes the movement of the first constraint upwards and parallel to itself, while this increase after point K has no effect on the feasible region and the improvement of the objective function. Because this increase will cause the first constraint being redundant, thus the value of the first source will increase until the constraint of this resource crosses the point K.

The first constraint will pass through the point K if the number on the right hand side is greater than 6. To calculate this increase, it is enough to place the coordinates of point K in the first constraint, which is as follows:

$$x_1 + 2x_2 = 3 + (2 \times 2) = 7$$

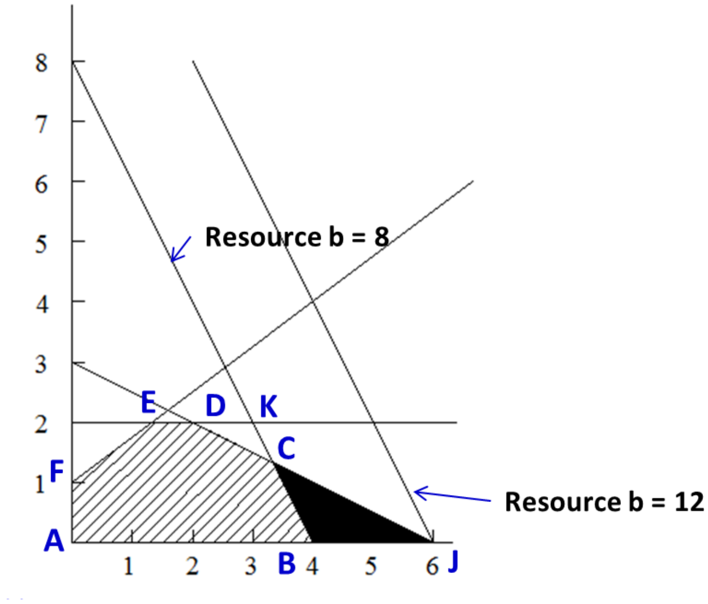
In this way, the maximum increase of the first limit is from 6 to 7, which is equal to one unit.

The figure below shows the effect of increasing the second source. This case has been made regardless of the changes of the first source. Increasing the second source until the second constraint reaches point J will improve Z (objective function). The coordinates of point J are obtained from the intersection of the second constraints and the non-negativity constraint of the x_2 variable as follows.

$$x_1 + 2x_2 = 6, x_2 = 0 \rightarrow x_1 = 6, x_2 = 0$$

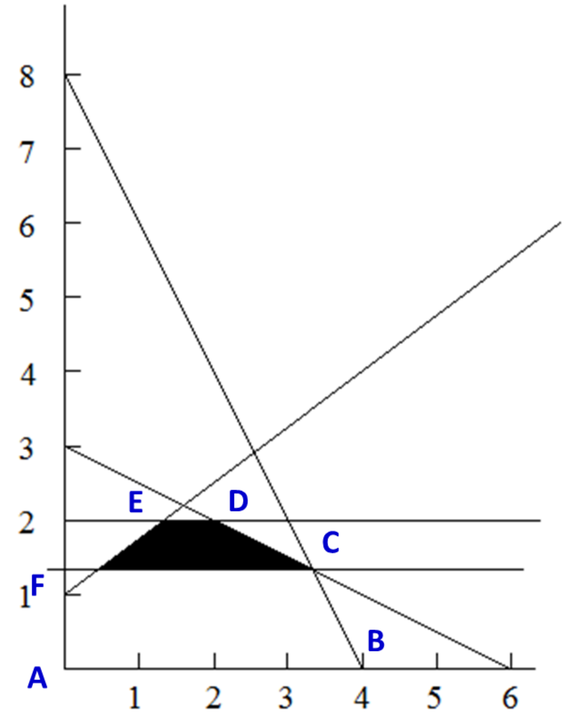
By placing the coordinates of point J in the second constraint, the number on the right hand side of the second constraint and the new value of source b are obtained.

$$2x_1 + x_2 = 1 \rightarrow 2(6) + (1)(0) = 12$$



In this way, the maximum increase of the right hand side of the second constraint is possible from 8 to 12, which will be 4 units.

Now check the decrease in the numbers on the right hand side of the redundant constriant. Since s_3 and s_4 are not zero, two constraints 3 and 4 are redundant. Consider constraint 4. The figure below shows that the fourth constraint (line ED) can be reduced as far as it passes point C, without affecting the optimal solution. Since the point C is $x_1 = 3\frac{1}{3}, x_2 = 1\frac{1}{3}$, the number on the right hand side of the fourth constraint can be reduced to $1\frac{1}{3}$ at most without a change in the optimal point.



Now consider the third constraint. Again, the right hand side of the constraint can be reduced so that the equation corresponding to the third constraint ($-x_1+x_2=1$) passes through point C. Therefore, the right hand side of the third constraint is equal to $-x_1+x_2=\left(-3\frac{1}{3}\right)+\left(1\frac{1}{3}\right)=-2$. This change does not affect the current optimal point, C. The results of the above discussions are summarized in the table below.

Resource	Type	Maximum change in Z value	Maximum change in resource value
1	Scarce	$13-12\frac{2}{3}=\frac{1}{3}$	$7-6=1$
2	Scarce	$18-12\frac{2}{3}=5\frac{1}{3}$	$12-8=4$
3	Redundant	$12\frac{2}{3}-12\frac{2}{3}=0$	$-2-1=-3$
4	Redundant	$12\frac{2}{3}-12\frac{2}{3}=0$	$1\frac{1}{3}-2=-\frac{2}{3}$

Which resources should be increased?

Considering the budget limitation, which we normally face in the economic aspect, we would like to know which of the sources has more priority in capital allocation. Naturally, because we want to increase profits, we are willing to invest in a source that increases profits the most. This is stated in the table below.

Resource	Type	Value of y_i
1	Scarce	$y_1 = \frac{1}{3}$
2	Scarce	$y_2 = \frac{4}{3}$
3	Redundant	$y_3 = 0$
4	Redundant	$y_4 = 0$

If y_i is the value of each unit of resource i, then y_i is obtained from the following formula.

$$y_i = \frac{Z^* \text{ Maximum change in optimum value}}{b_i \text{ Allowable increase in source i}}$$

Sensitivity analysis of objective function coefficients

The change in the coefficients of the objective function affects the slope of the objective function line. If this change exceeds a certain amount, the optimal point will change. This means that changes in the coefficients of the objective function can change a set of important restrictions and consequently the status (scarce or redundant) of the resources. The objective function sensitivity analysis aims to answer this question:

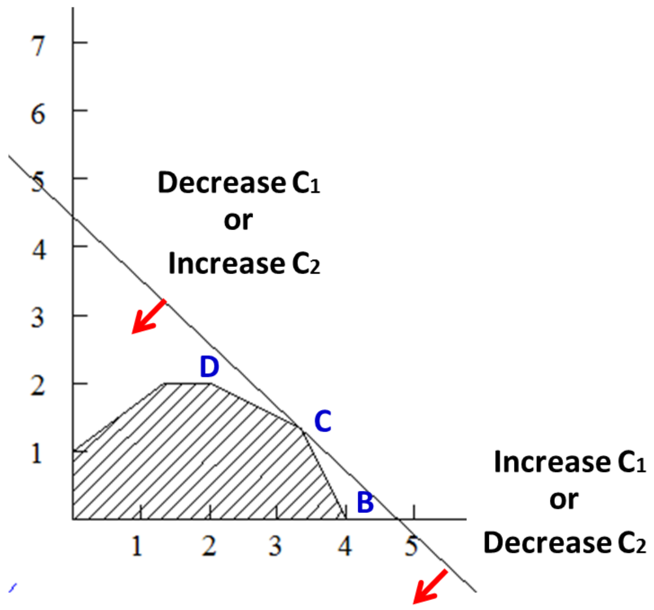
How much coefficients of the objective function can be changed (increased or decreased) without changing the optimal point?

Example:

If the coefficients of the objective function in the previous example are denoted by C_1 and C_2, the objective function becomes:

$$Z = C_1 x_1 + C_2 x_2$$

As shown in the figure below, as C_1 increases or C_2 decreases, the objective function line rotates clockwise around point C. Conversely, a decrease in C_1 or an increase in C_2 causes Z to move counterclockwise. Therefore, point C remains optimal until the slope of the objective function (Z) changes between the slopes of constraints 1 and 2. When the slope Z coincides with the slope of the first constraint, the problem will have two optimal corner points C and D. Similarly, when the slope of Z coincides with the slope of the second constraint, the two points C and B will be optimized. Any small change outside the above defined range for C_1 will place the new optimal solution at point B or D. In order to calculate the range of changes, first the numerical value of the coefficient x_2 is kept constant and the coefficient x_1 is denoted by C_1.



The figure above shows that C_1 can be increased until to the second constraint (CB line), or decreased until to the first constraint (DC line).

Therefore, the minimum or maximum value of C_1 can be obtained by making the slope of Z equal to the slope of the first and second constraint. The slope of Z is $-C_1/2$ and the slope of the first and second limits will be -1/2 and -2. Here, the minimum value of C_1 is found from the following relationship.

$$-\frac{C_1}{2} = \frac{-1}{2} \rightarrow C_1 = 1$$

Similarly, the maximum value of C_1 to remain optimal at point C is:

$$-\frac{C_1}{2} = \frac{-2}{1} \rightarrow C_1 = 4$$

The range of changes of C_1 to remain optimal at point C is as follows:

$$1 \le C_1 \le 4$$

When C_1 is equal to 1, the optimal solution will be at point C or D. If the value of C_1 becomes less than 1, the optimal solution moves to D. Similarly, there can be an interpretation for C_1 equal to or greater than 4. If C_1 is greater than 4, the optimal solution moves to B.

Added a new limit

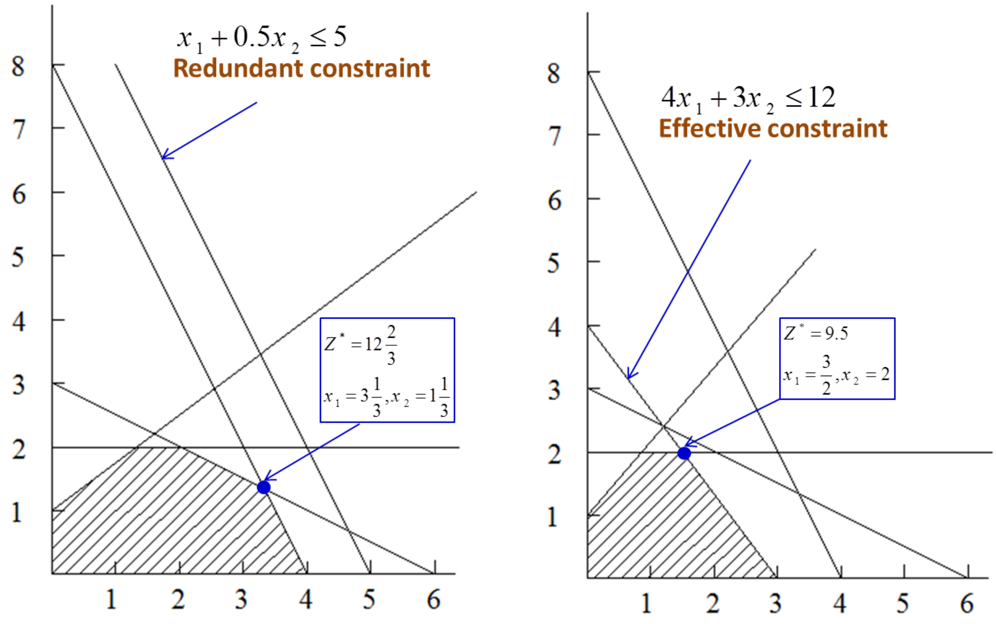
After solving the problem and finding the optimal solution, it is possible that due to economic or technical conditions, a new constraint will be added to the previous constraints, and may cause a decrease in the feasible region or do not have an impact on a feasible region. This effect can be shown in the following two cases:

1) Adding a redundant constraint

As stated earlier, a redundant constraint is a constraint whose presence or absence has no effect on the feasible region and consequently has no optimal solution. According to the previous example, as shown in the figure below, if a restriction of $x_1+0.5x_2<=5$ is added to the previous four constraints, it will not affect the feasible region and the optimal solution remains optimal.

2) Addition of an effective constraint

An effective constraint is a constraint that is effective in changing the feasible region and can change the optimal solution. The constraint of $4x_1+3x_2<=12$ reduces the feasible area and changes Z* from 12.66 to 9.5.



Parametric linear programming

In the sensitivity analysis, the discrete effect of model parameters on the final solution was investigated. Whenever the impact of continuous changes of parameters in all possible domains is considered, parametric linear programming should be used.

Systematic change of parameters c_j

Consider the following objective function:

$$Z=\sum_{j=1}^{n} c_j x_j$$

In parametric programming, the above objective function is replaced by the following function.

$$Z(\theta)=\sum_{j=1}^{n}(c_j+\alpha_j\theta)x_j$$

α_j are constant data that will represent the rate of changes in the coefficients of the objective function. The value of θ gradually becomes larger than zero. To illustrate the performance of the linear programming model as θ varies, consider the following example.

Example:

$$\begin{aligned}
&Max \quad Z=3x_1+5x_2\\
&s.t.\\
&(1)\quad x_1 + x_3 = 4\\
&(2)\quad 2x_2 + x_4 = 12\\
&(3)\quad 3x_1+2x_2 + x_5 = 18\\
&\quad x_i \geq 0 \quad i=1,\dots,5.
\end{aligned}$$

Solution:

Consider the value $\alpha_1=2$ and $\alpha_2=-1$. Therefore, the objective function is as follows.

$$Z(\theta)=(3+2\theta)x_1+(5-\theta)x_2$$

We start from the final simplex table with $\theta=0$, the objective function becomes as follows.

$$Z+1.5x_4+x_5=36$$

We add the changes of the objective function to the left side of the objective function, which is as follows.

$$Z-2\theta x_1+\theta x_2+1.5x_4+x_5=36$$

Because x_1 and x_2 are basic variables (which appeared in equations 2 and 3), the coefficient of these two variables in the above objective function should be equal to zero, which is possible by adding equations 2 and 3 to the objective function, which is as follows:

$$Z+(1.5-\frac{7}{6}\theta)x_4+(1+\frac{2}{3}\theta)x_5=36-2\theta$$

According to the stopping condition of the prime simplex algorithm, as long as the coefficients of the non-basic variables remain non-negative, the basic feasible solution is optimized, so we have:

$$1.5-\frac{7}{6}\theta\geq 0\rightarrow 0\leq\theta\leq\frac{9}{7}$$
$$1+\frac{2}{3}\theta\geq 0\rightarrow 0\leq\theta$$

Therefore, if $\theta>\frac{9}{7}$ becomes, x_4 is selected as the basic input variable and a new optimal solution is obtained using the prime simplex method. The new simplex table with x_4 entering the base and x_3 leaving the base (according to the ratio test) is as follows.

Range of θ	Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS
$\frac{9}{7}\leq\theta\leq 5$	Z	0	1	0	0	$\frac{-9+7\theta}{2}$	0	$\frac{5-\theta}{2}$	$27+5\theta$
	X_4	1	0	0	0	3	1	-1	6
	X_2	2	0	0	1	-1.5	0	0.5	3
	X_1	3	0	1	0	1	0	0	4

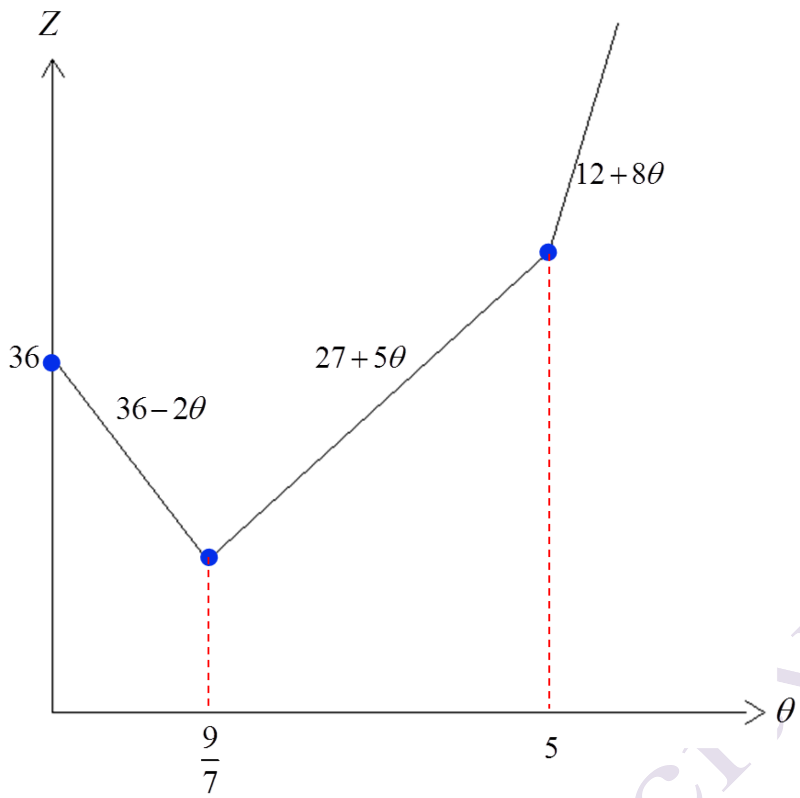
Therefore, if $\theta>5$ becomes, x_5 is selected as the basic input variable and a new optimal solution is obtained using the initial simplex method. The new simplex table with x_5 entering the base and x_2 leaving the base (according to the ratio test) is as follows.

Range of θ	Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS
$\theta\geq 5$	Z	0	1	0	$-5+\theta$	$3+2\theta$	0	0	$12+8\theta$
	X_4	1	0	0	2	0	1	0	12
	X_5	2	0	0	2	-3	0	1	6
	X_1	3	0	1	0	1	0	0	4

As it is clear in the above table, zero rows will be non-negative for $\theta > 5$ and the value of θ can be increased to infinity and there will be no change in the basic variables. The summary of the above procedure for all θ values is given in the table below.

Range of θ	Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS
$0 \le \theta \le \frac{9}{7}$	Z	0	1	0	0	0	$\frac{9-7\theta}{6}$	$\frac{3+2\theta}{3}$	$36-2\theta$
	X_3	1	0	0	0	1	0.33	-0.5	2
	X_2	2	0	0	1	0	0.5	0	6
	X_1	3	0	1	0	0	-0.33	0.33	2
$\frac{9}{7} \le \theta \le 5$	Z	0	1	0	0	$\frac{-9+7\theta}{2}$	0	$\frac{5-\theta}{2}$	$27+5\theta$
	X_4	1	0	0	0	3	1	-1	6
	X_2	2	0	0	1	-1.5	0	0.5	3
	X_1	3	0	1	0	1	0	0	4
$\theta \ge 5$	Z	0	1	0	$-5+\theta$	$3+2\theta$	0	0	$12+8\theta$
	X_4	1	0	0	2	0	1	0	12
	X_5	2	0	0	2	-3	0	1	6
	X_1	3	0	1	0	1	0	0	4

Also, the value of the objective function of the optimal solution as a function of θ is as follows.



Systematic changes of the right parameters (b_i)

In this case, b_i is replaced by $b_i + \alpha_i \theta$, where α_i are fixed data. Therefore, the problem is as follows.

$$Max\ \ Z(\theta) = \sum_{j=1}^{n} c_j x_j$$

$s.t.$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + \alpha_i \theta \quad i = 1, \ldots, m$$

$$x_j \ge 0 \quad j = 1, \ldots, n.$$

The purpose of this section is to determine the optimal job as a function of θ. The solution procedure described below is very similar to the process of what was said for the change of c_j, which is because the change in the coefficients of the objective function of the prime problem is equivalent to the change in the coefficients of the right hand side of the dual problem. To explain the issue further, we will solve some examples.

Example:

Use the parametric linear programming procedure to perform systematic changes in b_i and obtain the optimal solution of the following problem as a function of θ for $0 \le \theta \le 25$.

$$Max \quad Z(\theta) = 2x_1 + x_2$$
$$s.t.$$
$$(1) \quad x_1 \le 10 + 2\theta$$
$$(2) \quad x_1 + x_2 \le 25 - \theta$$
$$(3) \quad x_2 \le 10 + 2\theta$$
$$x_i \ge 0 \quad i = 1, 2.$$

Solution:

Regardless of the value of θ, we get the optimal solution of the above model, which is given in the table below.

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS
Z	0	1	-2	-2	0	0	0	0
X_3	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	1	1	0	1	0	$25-\theta$
X_5	3	0	0	1	0	0	1	$10+2\theta$
Z	0	1	0	-1	2	0	0	$20+4\theta$
X_1	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	0	1	-1	1	0	$15-3\theta$
X_5	3	0	0	1	0	0	1	$10+2\theta$
Z	0	1	0	0	2	0	1	$30+6\theta$
X_1	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	0	0	-1	1	-1	$5-5\theta$
X_2	3	0	0	1	0	0	1	$10+2\theta$

For $0 \le \theta \le 1$, the last table will have optimal conditions. But for $\theta > 1$, variable X_4 leaves the base and variable X_5 enters the base. In the next optimal table, for $1 \le \theta \le 5$ the optimal table will remain. But for $\theta > 5$, variable X_2 leaves the base and variable X_3 enters the base. In the next optimal table, for $\theta \ge 5$, the optimal table will remain. As θ increases, there will be no change in the optimal conditions and feasibility of the table. The corresponding simplex table is given for each change of θ.

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS
Z	0	1	-2	-2	0	0	0	0
X_3	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	1	1	0	1	0	$25-\theta$
X_5	3	0	0	1	0	0	1	$10+2\theta$
Z	0	1	0	-1	2	0	0	$20+4\theta$
X_1	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	0	1	-1	1	0	$15-3\theta$
X_5	3	0	0	1	0	0	1	$10+2\theta$
Z	0	1	0	0	2	0	1	$30+6\theta$
X_1	1	0	1	0	1	0	0	$10+2\theta$
X_4	2	0	0	0	-1	1	-1	$5-5\theta$
X_2	3	0	0	1	0	0	1	$10+2\theta$
Z	0	1	0	0	1	1	0	$35+\theta$
X_1	1	0	1	0	1	0	0	$10+2\theta$
X_5	2	0	0	0	1	-1	1	$5\theta-5$
X_2	3	0	0	1	-1	1	0	$15-3\theta$
Z	0	1	0	1	0	2	0	$50-2\theta$
X_1	1	0	1	1	0	1	0	$25-\theta$
X_5	2	0	0	1	0	0	1	$10+2\theta$
X_3	3	0	0	-1	1	-1	0	$3\theta-15$

The results are summarized in the table below.

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \le \theta \le 1$	$(10+2\theta, 10+2\theta)$	$30+6\theta$
$1 \le \theta \le 5$	$(10+2\theta, 15-3\theta)$	$35+\theta$
$5 \le \theta \le 25$	$(25-\theta, 0)$	$50-2\theta$

Example:

$$Max \quad Z(\theta) = 8x_1 + 24x_2$$
$$s.t.$$
$$(1) \qquad x_1 + 2x_2 \le 10$$
$$(2) \qquad 2x_1 + x_2 \le 10$$
$$x_i \ge 0 \quad i = 1, 2.$$

Suppose that $Z(\theta)$ represents the profit and the objective function can be changed to some extent by the proper transfer of manpower between the two activities. Specifically, suppose that the profit of the first activity can be increased from 8 (up to 18). But for every unit increase in the profit of the first unit, the profit of the second activity decreases by two units. Therefore, $Z(\theta)$ should be represented as follows.

$$Z(\theta) = (8+\theta)x_1 + (24-2\theta)x_2$$

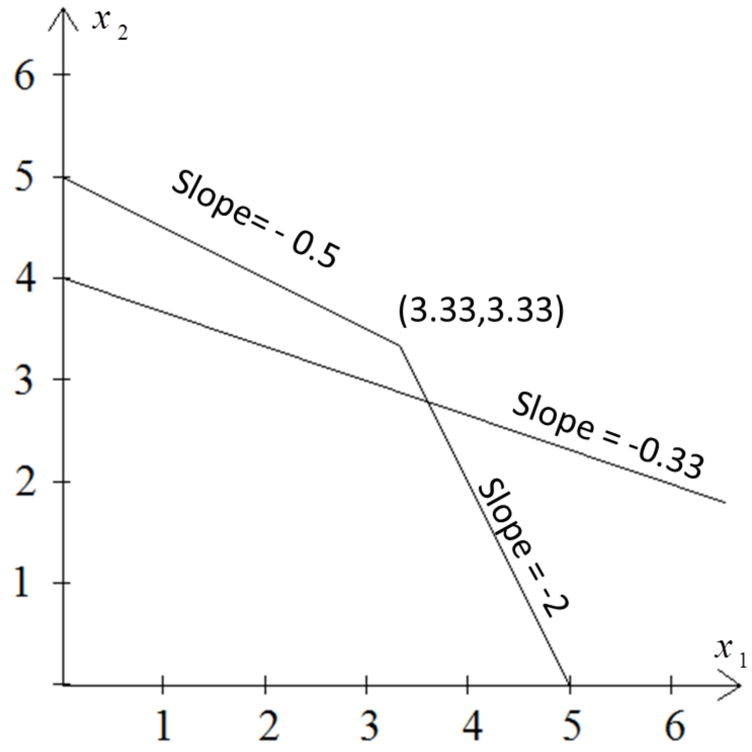
that θ itself is a decision variable so that $0 \le \theta \le 10$.

a) Use parametric linear programming and determine the optimal solution as well as the optimal value of $Z(\theta)$ as a function of θ for $0 \le \theta \le 10$.

b) Determine the optimal value of θ. Then show how to find this optimal value using only the solution of two linear programming problems.

Solution:

a)



It can be concluded from the above figure:

The optimal solution (0,5) holds as long as

$$-\frac{8+\theta}{24-2\theta} \geq -0.5 \rightarrow \theta \leq 2$$

The optimal solution (3.33,3.33) holds as long as

$$-0.5 \geq -\frac{8+\theta}{24-2\theta} \geq -2 \rightarrow 2 \leq \theta \leq 8$$

The optimal solution (5,0) holds as long as

$$-\frac{8+\theta}{24-2\theta} \leq -2 \rightarrow \theta \geq 8$$

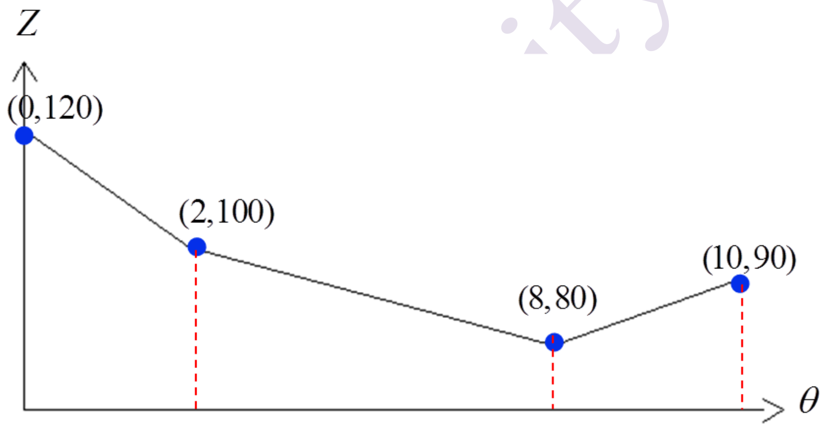
If we want to get the above results using the simplex table, we do as follows. At iteration 0, X_3 leaves the base and X_2 enters the base, which in this case has optimal conditions for $\theta = 0$ in the iteration 1. For $0 \le \theta \le 2$, the iteration 1 has optimality conditions. If $\theta > 2$ is, the coefficient of X_1 becomes negative in the row 0, and therefore, according to the prime simplex method, X_1 enters the base and the variable X_4 leaves the base. The row 0 coefficients for $2 \le \theta \le 8$ will be non-negative in iteration 2. For $\theta > 8$, the coefficient X_3 becomes negative in the row 0 and enters the basis according to the prime simplex method, and the variable X_2 leaves the base, which leads to iteration 3. In the iteration 3, the non-negative zero row will remain for $\theta \ge 8$ and the basic solution will not change with the increase of this parameter. For different values of θ, we calculate the value of the objective function using the simplex table.

Iteration	Basic variable	Row	Z	X_1	X_2	X_3	X_4	RHS
0	Z	0	1	$-8-\theta$	$-24+2\theta$	0	0	0
	X_3	1	0	1	2	1	0	10
	X_4	2	0	2	1	0	1	10
1	Z	0	1	$4-2\theta$	0	$12-\theta$	0	$120-10\theta$
	X_2	1	0	0.5	1	0.5	0	5
	X_4	2	0	1.5	0	-0.5	1	5
2	Z	0	1	0	0	$\frac{40-5\theta}{3}$	$\frac{4\theta-8}{3}$	$\frac{320-10\theta}{3}$
	X_2	1	0	0	1	0.66	-0.33	3.33
	X_1	2	0	1	0	-0.33	0.66	3.33
3	Z	0	1	0	$\frac{-40+5\theta}{2}$	0	$\frac{8+\theta}{2}$	$40+5\theta$
	X_3	1	0	0	1.5	1	-0.5	5
	X_1	2	0	1	0.5	0	0.5	5

The table below shows the optimal solution of the model for all values of $0 \le \theta \le 10$.

θ	(x_1^*, x_2^*)	$Z^*(\theta)$
$0 \le \theta \le 2$	$(0,5)$	$120-10\theta$
$2 \le \theta \le 8$	$(3.33,3.33)$	$\frac{(320-10\theta)}{2}$
$8 \le \theta \le 10$	$(5,0)$	$40+5\theta$

The above table is displayed graphically as follows.



b) According to the graph above, we can understand that we will have the best model solution for $\theta = 0$. Considering that $Z(\theta)$ is a convex function in terms of θ, the maximum value occurs at the boundary of θ and therefore it is enough to solve the linear programming for the values of $\theta = 0$ and $\theta = 10$.

Example:

Obtain the optimal value of the following model for $0 \le \theta \le 20$ using parametric linear programming.

$$Max \quad Z(\theta) = (20+4\theta)x_1 + (30-3\theta)x_2 + 5x_3$$

$s.t.$

(1) $3x_1 + 3x_2 + x_3 \leq 10$

(2) $8x_1 + 6x_2 + 4x_3 \leq 25$

(3) $6x_1 + x_2 + x_3 \leq 15$

$x_i \geq 0 \quad i = 1,2,3.$

Solution:

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
Z	0	1	$-20-4\theta$	$-30+3\theta$	-5	0	0	0	0
X_4	1	0	3	3	1	1	0	0	10
X_5	2	0	8	6	4	0	1	0	25
X_6	3	0	6	1	1	0	0	1	15
Z	0	1	$10-7\theta$	0	$5-\theta$	$10-\theta$	0	0	$100-10\theta$
X_2	1	0	1	1	0.33	0.33	0	0	3.33
X_5	2	0	2	0	2	-2	1	0	5
X_6	3	0	5	0	0.66	-0.33	0	1	11.66
Z	0	1	0	0	$\frac{55-\theta}{15}$	$\frac{160-22\theta}{15}$	0	$\frac{-10+7\theta}{5}$	$\frac{230+19\theta}{3}$
X_2	1	0	0	1	0.2	0.4	0	-0.2	1
X_5	2	0	0	0	26/15	-28/15	1	-0.4	0.33
X_1	3	0	1	0	2/15	-1/15	0	1/5	2.33
Z	0	1	0	$\frac{-80+11\theta}{3}$	$\frac{-5+2\theta}{3}$	0	0	$\frac{10+2\theta}{3}$	$50+10\theta$
X_4	1	0	0	2.5	0.5	1	0	-0.5	2.5
X_5	2	0	0	14/3	8/3	0	1	-4/3	5
X_1	3	0	1	1/6	1/6	0	0	1/6	2.5

The optimal value of the objective function for different values of θ is as follows.

θ	(x_1^*, x_2^*, x_3^*)	$Z^*(\theta)$
$0 \le \theta \le \frac{10}{7}$	$(0, \frac{10}{3}, 0)$	$100 - 10\theta$
$\frac{10}{7} \le \theta \le \frac{80}{11}$	$(3.33, 1, 0)$	$\frac{(230 + 19\theta)}{3}$
$\frac{80}{11} \le \theta$	$(2.5, 0, 0)$	$50 + 10\theta$

Example:

Use the parametric linear programming method and obtain the optimal solution of the following problem as a function of θ for $0 \le \theta \le 30$.

$$\begin{aligned} &Max \quad Z(\theta) = 5x_1 + 42x_2 + 28x_3 + 49x_4 \\ &s.t. \\ &(1) \qquad 3x_1 - 2x_2 + x_3 + 3x_4 \le 135 - 2\theta \\ &(2) \qquad 2x_1 + 4x_2 - x_3 + 2x_4 \le 78 - \theta \\ &(3) \qquad x_1 + 2x_2 + x_3 + 2x_4 \le 30 + \theta \\ &\qquad x_i \ge 0 \quad i = 1, .., 4. \end{aligned}$$

Solution:

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
Z	0	1	-5	-42	-28	-49	0	0	0	0
X_5	1	0	3	-2	1	3	1	0	0	$135 - 2\theta$
X_6	2	0	2	4	-1	2	0	1	0	$78 - \theta$
X_7	3	0	1	2	1	2	0	0	1	$30 + \theta$
Z	0	1	19.5	7	-3.5	0	0	0	24.5	$735 + 24.5\theta$
X_5	1	0	1.5	-5	-0.5	0	0	0	-1.5	$90 - 3.5\theta$
X_6	2	0	1	2	-2	0	0	0	-1	$48 - 2\theta$
X_4	3	0	0.5	1	0.5	1	0	0	0.5	$15 + 0.5\theta$
Z	0	1	24	14	0	7	0	0	28	$840 + 28\theta$
X_5	1	0	2	-4	0	1	0	0	-1	$105 - 3\theta$
X_6	2	0	3	6	0	4	0	0	0.5	108
X_3	3	0	1	2	1	2	0	0	1	$30 + \theta$

For $\theta \leq 30$, the optimal solution is $840 + 28\theta$ and $(0, 0, 30 + \theta, 0, 105 - 3\theta, 108)$.